1. Let $x_n := \frac{1}{1^2} + \frac{1}{2^2} + \dots + \frac{1}{n^2}$ for each $n \in \mathbb{N}$.

Show that (x_n) is convergent

2. (a) State the definition of Cauchy sequence

(b) Let $x_n := \sqrt{n}$ for $n \in \mathbb{N}$. Show directly from the definition that (x_n) is not a Cauchy sequence

3. Let (x_n) be a contraction sequence of real numbers,

i.e. $\exists\, c \in (0,1)$ s.t. $|x_{n+2} - x_{n+1}| \le c\,|x_{n+1} - x_n| \quad \forall n \in \mathbb{N}$

(a) Show that (x_n) is Cauchy and hence convergent

(b) Let $x_1 := 2$ and $x_{n+1} := 2 + \frac{1}{x_n}$ for $n \ge 1$.

Show that (x_n) is contractive. Find the limit

4. (a) Let (x_n) be a sequence of real numbers.

State the definition of properly divergent sequence: $\lim (x_n) = +\infty$ or $\lim (x_n) = -\infty$

(b) Show that the sequence $(\sqrt{n^2+2})$ is properly divergent.

(c) Show that if $\lim \left(\frac{a_n}{n}\right) = L$, $L > 0$, then $\lim (a_n) = +\infty$

1. F_1 : (x_n) is monotone increasing sequence.

F_2 : (x_n) is bounded above.

If F_1 and F_2 are true, By Monotone Convergence Theorem, x_n is convergent.

Pf of F_1 : $x_{n+1} = x_n + \frac{1}{(n+1)^2} \Rightarrow x_{n+1} \geq x_n$

Pf of F_2 : $x_n = \frac{1}{1^2} + \frac{1}{2^2} + \dots + \frac{1}{n^2}$

$$\leq 1 + \frac{1}{1\times 2} + \frac{1}{2\times 3} + \dots + \frac{1}{n\times(n+1)}$$

$$\leq 1 + 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \dots + \cancel{\frac{1}{n}} - \frac{1}{n+1}$$

$$\leq 2 - \frac{1}{n+1}$$

$$\leq 2 \qquad \forall n \in \mathbb{N}$$

Then (x_n) is bounded.

2(a) A sequence $X = (x_n)$ of $\mathbb{R}$ is said to be Cauchy sequence if for every $\varepsilon > 0$, there exist $H(\varepsilon)$ s.t. for all natural number $n, m \geq H(\varepsilon)$, $|x_n - x_m| < \varepsilon$.

(b) negation of Cauchy sequence:

$\exists\, \varepsilon_0 > 0$ s.t. $\forall H \in \mathbb{N}$, $\exists\, n, m \in \mathbb{N}$ s.t. $n, m \geq H$ and $|x_n - x_m| \geq \varepsilon_0$

if $n > m$, choose $n = 4m$, $\sqrt{n} - \sqrt{m} = \sqrt{4m} - \sqrt{m}$

$$= \sqrt{m}$$

$$> \frac{1}{2}$$

Then (x_n) is not Cauchy sequence.

3. (a) $$|x_{n+2} - x_{n+1}| \le c\,|x_{n+1} - x_n|$$
$$\le c^2\,|x_n - x_{n-1}|$$
$$\vdots$$
$$\le c^n\,|x_2 - x_1|$$

Then check (x_n) is Cauchy

if $m > n$, $$|x_m - x_n| \le |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \dots + |x_{n+1} - x_n|$$
$$\le (c^{m-2} + c^{m-3} + \dots + c^{n-1})\,|x_2 - x_1|$$
$$= c^{n-1}\left(\frac{1 - c^{m-n}}{1-c}\right)|x_2 - x_1|$$
$$\le c^{n-1}\left(\frac{1}{1-c}\right)|x_2 - x_1| \quad (\because c^{m-n} > 0)$$

Since $0 < c < 1$, $\lim c^n = 0$

Let $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. if $n \ge N$, $|c^n| < \varepsilon \cdot c\,(1-c)\,\frac{1}{|x_2 - x_1|}$

Choose this N, if $m, n \ge N$

$$|x_m - x_n| \le c^n \cdot \frac{1}{c\cdot(1-c)}\,|x_2 - x_1| < \varepsilon$$

Hence, (x_m) is Cauchy $\Rightarrow$ (x_n) is convergent

($\because$ Cauchy Convergence Criterion)

(b) By MI, $2 \le x_n \le 3 \quad \forall n \in \mathbb{N}$

$$|x_{n+2} - x_{n+1}| = \left|2 + \frac{1}{x_{n+1}} - 2 - \frac{1}{x_n}\right|$$
$$= \left|\frac{x_n - x_{n+1}}{x_{n+1}\,x_n}\right|$$
$$= \frac{1}{|x_{n+1}|}\,\frac{1}{|x_n|}\,|x_{n+1} - x_n|$$
$$\le \frac{1}{4}\,|x_{n+1} - x_n|$$

Then (x_n) is contractive.

Hence (x_n) is convergent

3 (b) let x be the limit of (x_n)

$\lim x_n = x$

$$x_{n+1} = 2 + \frac{1}{x_n}$$

$$\Rightarrow \lim x_{n+1} = 2 + \frac{1}{\lim x_n}$$

$$\Rightarrow x = 2 + \frac{1}{x}$$

$$\Rightarrow x^2 - 2x - 1 = 0$$

$$\Rightarrow x = 1 \pm \sqrt{2}$$

Since $2 \le x_n \le 3 \Rightarrow 2 \le x \le 3 \Rightarrow x = 1 + \sqrt{2}$

4 (a) $\lim (x_n) = +\infty$ $(-\infty)$ if for every $\alpha \in \mathbb{R}$,

there exist $K(\alpha) \in \mathbb{N}$ s.t. if $n \ge K(\alpha)$, $x_n > \alpha$ $(x_n < \alpha)$

(b) $\sqrt{n^2+2} \ge \sqrt{n^2} = n$

Let $\alpha \in \mathbb{R}$, Choose $K \in \mathbb{N}$ s.t. $K > \alpha$,

if $n \ge K$, $\sqrt{n^2+2} \ge n \ge K > \alpha$

Then $\lim \sqrt{n^2+2} = +\infty$

(c) Let $\varepsilon = \frac{L}{2}$, $\exists\, N \in \mathbb{N}$ s.t. if $n \ge N$, $\left|\frac{a_n}{n} - L\right| < \varepsilon$

$$\Rightarrow \left|\frac{a_n}{n} - L\right| < \frac{L}{2}$$

$$\Rightarrow -\frac{L}{2} < \frac{a_n}{n} - L < \frac{L}{2}$$

$$\Rightarrow \frac{L}{2} < \frac{a_n}{n} < \frac{3L}{2}$$

$$\Rightarrow \frac{L}{2} n < a_n < \frac{3L}{2} n$$

Let $\alpha \in \mathbb{R}$, choose $N' \in \mathbb{N}$ s.t. $N' > \alpha \cdot \frac{2}{L}$

Choose $K := \max\{N, N'\}$, if $n \ge K$, $a_n > \frac{L}{2} n > \frac{L}{2} N' > \alpha$

Then $\lim a_n = +\infty$